Fractional Reduced Differential Transform Method for Belousov-Zhabotinsky Reaction Model

Aung Zaw Myint[*]

Abstract

In this paper, Belousov-Zhabotinsky (B-Z) reaction model with Caputo fractional time derivative is investigated by the fractional reduced differential transform method (FRDTM) methods, an iterative technique. The outcome using FRDTM method reveals an efficiency with high accuracy and minimal computations for numerical solutions. Moreover, the solution profiles which demonstrate the behavior of the obtained result are presented.

Keywords: Belousov-Zhabotinskii system, Fractional reduced differential transform method.

Preliminaries

The applications of fractional calculus have been established in various connected bifurcation of science and engineering such as found in quantum mechanics (Joseph et al., 2012), random walk (Hilfer and Anton, 1995), astrophysics (Tarasov, 2006), chaos theory (Baleanu et al., 2017), electrodynamics (Nasrolahpour, 2013), viscoelasticity (Mainardi, 2010), nanotechnology (Baleanu, et al., 2010) and other field (Chen, et al., 2013). In the twentieth century, (Caputo, 1969, Liao, 1998, Podlubny, 1999, Miller and Ross, 1993) have described the essential properties of fractional calculus. In this present work, a nonlinear oscillatory system called the Belousov-Zhabotinskii, (B-Z) with Caputo fractional time derivative is investigated by the fractional reduced differential transform method (FRDTM) methods, an iterative technique. The B-Z is a family of oscillating chemical reactions and is interesting because this reaction is a chemical reaction which can demonstrate both temporal oscillations and spatial traveling concentration waves that are accompanied with dramatic color changes (Gibbs, 1980). The simplified Noyes-Field fractional model for the B-Z reaction is given as (Ye, et al., 1987):

$$\begin{cases} \dfrac{\partial^{u} p}{\partial t^{u}} = \psi_1 \dfrac{\partial^2 p}{\partial t^2} + \beta\xi q + p - p^2 - \xi pq, & 0 < \mu \le 1, \\ \dfrac{\partial^{u} q}{\partial t^{u}} = \psi_2 \dfrac{\partial^2 p}{\partial t^2} + \gamma q - \lambda pq, & \\ p(x,0) = g_1(x), & \\ q(x,0) = g_2(x), & x \in R,\ t \ge 0, \end{cases} \tag{1}$$

where ψ_1 and ψ_2 are the diffusing constants for the concentration p and q respectively, γ and β are constants, ξ and $\lambda \neq 1$are positive parameters and μ is the fractional order.

Fractional nonlinear problems are often more difficult to solve because its operator is defined by integral. However, different computational schemes are developed and have been used to investigate both the exact and numerical solution of these fractional problems. Some of the used methods are the Adomian decomposition method (ADM) (Adomian, 1994, Mainardi, 2010), variational iteration method, (VIM) (Das, 2009, He, 1998), homotopy perturbation

[*] Lecturer, Dr., Department of Mathematics, University of Mandalay

method (HPM) (He, 1999, He, 2003)), residual power series method, (RPSM) (Senol, et al., 2019), Sumudu decomposition method (SDM) (Eltayeb and Kilicman, A 2012), homotopy analysis method, (HAM) (Liao, 2004), Laplace decomposition method, (LDM) (Khuri, 2001, Jaradat et al., 2018). These methods have used modified version of generalized Taylor power series method to construct numerical solution to the time-fractional B-Z system for the case when the diffusing constants $\psi_1 = \psi_2 = 1$ and the parameters $\gamma = \beta = 0$.

This paper is organized as follows. In Section 1, we present basic definitions and preliminaries of B-Z reaction model. In Section 2, we state the analysis of time-fractional Belousov-Zhabotinsky reaction (TFB-Z) model. In Section 3, we give the solution for time-fractional Belousov-Zhabotinsky (TFB-Z) system of equations and present the solution profiles which demonstrate the behavior of the obtained results.

Definition

The Riemann-Liouville, (R-L) fractional integral of order $\mu(\mu \ge 0)$ of a function $Q(x,t) \in C_m,\ \ m \ge -1,$ is given as (Dhaigude and Nikam , 2012, Kilbas, et al., 2006),

$$J^{\mu}p(x,t) = \frac{1}{\Gamma(\mu)}\int_0^t (t-\varepsilon)^{\mu-1}p(x,\varepsilon)d\varepsilon, \qquad \mu, t > 0, \tag{2}$$

where Γ denotes the classical gamma function and $J^0 p(x,t) = p(x,t)$. For example,

$$J^{\mu}t^{\alpha} = \frac{\Gamma(\mu+1)}{\Gamma(\mu+1+\alpha)}t^{\mu+\alpha}. \tag{3}$$

Definition

In the Caputo's sense, the fractional derivative of $p(x,t)$ (denoted by $D^{u}p(x,t)$) for $\varphi - 1 < \mu < \varphi, \varphi \in N$ is defined as (Kilbas, et al., 2006),

$$D^{\mu}p(x,t) = \begin{cases} p^{(\varphi)}(x,t), & \mu = \varphi, \\ J^{\varphi-\mu}p^{(\varphi)}, & \varphi - 1 < \mu < \varphi, \end{cases} \tag{4}$$

where

$$J^{\varphi-\mu}p^{(\varphi)}(x,t) = \frac{1}{\Gamma(\varphi-\mu)}\int_0^t (t-\varepsilon)^{\varphi-\mu-1}p^{(\varphi)}(x,\varepsilon)d\varepsilon, \qquad \mu, t > 0. \tag{5}$$

Definition

The Laplace transform, (LT) of a Caputo fractional derivative is given as (Das, 2009, Kilbas, et al., 2006),

$$L[D_t^{\mu}p(x,t)] = s^{\mu}L[p(x,t)] - \sum_{m=0}^{n-1} s^{\mu-m-1}p^{m}(x,0^{+}), \quad n-1 < \mu \le n. \tag{6}$$

Analysis of the Proposed Method

Consider the time-fractional Belousov-Zhabotinsky, (TFB-Z) (1) with the diffusing constants for the concentration, $\Psi_1 = \Psi_2 = 1$,

$$\begin{cases} \dfrac{\partial^{\mu} p}{\partial t^{\mu}} = \dfrac{\partial^2 p}{\partial t^2} + \beta\xi q + p - p^2 - \xi pq, \qquad 0 < \mu \le 1, \\ \dfrac{\partial^{\mu} q}{\partial t^{\mu}} = \dfrac{\partial^2 p}{\partial t^2} + \gamma q - \lambda pq, \\ p(x,0) = g_1(x), \\ q(x,0) = g_2(x). \end{cases} \tag{7}$$

Analysis of FRDTM

The functions $p(x,t)$ and $q(x,t)$ are analytic and continuously differentiated in the interested domain. In regard to the properties of differential transform, functions $p(x,t)$ and $q(x,t)$ can be expressed as

$$p(x,t) = \sum_{m=0}^{\infty} P_m(x) t^{m\mu}, \qquad q(x,t) = \sum_{m=0}^{\infty} Q_m(x) t^{m\mu}, \tag{8}$$

where

$$P_m(x) = \frac{1}{\Gamma(m\mu+1)} \left[\frac{\partial^{m\mu} p(x,t)}{\partial t^{m\mu}} \right]_{t=0}, \qquad Q_m(x) = \frac{1}{\Gamma(m\mu+1)} \left[\frac{\partial^{m\mu} p(x,t)}{\partial t^{m\mu}} \right]_{t=0}. \tag{9}$$

Here, μ is the fractional order and the t-dimensional spectrum functions $P_k(x)$ and $Q_k(x)$ are respectively the transformed functions of $p(x,t)$ and $q(x,t)$. According to Table 1, the iteration formulas for (7) is

$$\begin{cases} \dfrac{\Gamma(m\mu+\mu+1)}{m\mu+1} P_{(m+1)}(x) = \dfrac{\partial^2 P_m}{\partial t^2} + \beta\xi Q_m + P_m - \sum_{r=0}^{m} P_r P_{(m-r)} - \xi \sum_{r=0}^{m} P_r Q_{(m-r)}, \\ \dfrac{\Gamma(m\mu+\mu+1)}{m\mu+1} Q_{(m+1)}(x) = \dfrac{\partial^2 Q_m}{\partial t^2} + \gamma Q_m - \lambda \sum_{r=0}^{m} P_r Q_{(m-r)}. \end{cases} \tag{10}$$

From initial condition (8), we write

$$p_0 = p(x,0), \qquad Q_0 = q(x,0). \tag{11}$$

Substituting (11) into (10), we obtain the $P_m(x)$ and $Q_m(x)$ values. The inverse transformation of the sets $\{P_m(x)\}_{m=0}^{k}$ and $\{Q_m(x)\}_{m=0}^{k}$ are respectively

$$P^{(k)}(x,t) = \sum_{m=0}^{k} P_m(x) t^{m\mu}, \qquad Q^{(k)}(x,t) = \sum_{m=0}^{k} Q_m(x) t^{m\mu}, \tag{12}$$

and

$$p(x,t)=\lim_{k\to\infty}P^{(k)}(x,t)=\sum_{m=0}^{\infty}P_m(x)t^{m\mu},\qquad q(x,t)=\lim_{k\to\infty}Q^{(k)}(x,t)=\sum_{m=0}^{\infty}Q_m(x)t^{m\mu},\tag{13}$$

which gives the exact solution of (7).

Table 1: The essential operations of FRDTM

Functional Form	**Transformed function**
p	$P_m=\dfrac{1}{\Gamma(m\mu+1)}\left[\dfrac{\partial^{m\mu}p}{\partial t^{m\mu}}\right]_{t=0}$
$w=\alpha_1 p+\alpha_2 q$	$w_m=\alpha_1 P_m+\alpha_2 Q_m$
$w=pq$	$w_m=\sum_{t=0}^{m}P_tQ_{(m-t)}=\sum_{t=0}^{m}P_tQ_{(m-t)}$
$w=D_t^{r\mu}p$	$W_m=\dfrac{\Gamma(m\mu+r\mu+1)}{\Gamma(m\mu+1)}P_{(m+r)}$
$w=\dfrac{\partial^{\nu}p}{\partial x^{\nu}}$	$W_m=\dfrac{\partial^{\nu}P_m}{\partial x^{\nu}}$
$w=x^rt^s$	$W_m=x^r\delta(m-s),\quad \delta(m-s)=\begin{cases}1, & m=s\\ 0, & m\neq s\end{cases}$
$w=e^{mt}$	$W_m=\dfrac{\lambda^m}{m!}$

Solution for TFB-Z system of Equations

In this section, application of the FRDTM proposed method to the time-fractional Belousov-Zhabotinsky (TFB-Z) system of equations is presented.

Example 1

Consider the nonlinear TFB-Z system at $\gamma=\beta=0$, then (7) reduced to

$$\begin{cases}\dfrac{\partial^{\mu}p}{\partial t^{\mu}}=\dfrac{\partial^{2}p}{\partial t^{2}}+\beta\xi q+p-p^2-\xi pq, & 0<\mu\leq 1,\\ \dfrac{\partial^{\mu}q}{\partial t^{\mu}}=\dfrac{\partial^{2}p}{\partial t^{2}}-\lambda pq,\end{cases}\tag{14}$$

with the initial condition

$$p(x,0)=\frac{1}{\left(e^{\sqrt{\frac{\lambda}{e}}x}+1\right)^2},\qquad q(x,0)=\frac{(1-\lambda)e^{\sqrt{\frac{\lambda}{e}}x}\left(e^{\sqrt{\frac{\lambda}{e}}x}+2\right)}{\xi\left(e^{\sqrt{\frac{\lambda}{e}}x}+1\right)^2}. \tag{15}$$

The exact solution of (14) when $\mu=1$ is

$$p(x,t)=\frac{e^{\frac{5\lambda}{3}t}}{\left(e^{\sqrt{\frac{\lambda}{6}}x}+e^{\frac{5\lambda}{3}t}\right)^2},\qquad q(x,0)=\frac{(1-\lambda)e^{\sqrt{\frac{\lambda}{6}}x}\left(e^{\sqrt{\frac{\lambda}{e}}x}+2e^{\frac{5\lambda}{3}t}\right)}{\xi\left(e^{\sqrt{\frac{\lambda}{e}}x}+e^{\frac{5\lambda}{3}t}\right)^2}. \tag{16}$$

Here $\lambda\neq 1$is a positive parameter.

■

FRDTM Solution:

From (10) with $\gamma=\beta=0$, we have

$$\begin{aligned}P_{(m+1)}(x)&=\frac{\Gamma(m\mu+\mu+1)}{\Gamma(m\mu+1)}\left\{\frac{\partial^2 P_m}{\partial t^2}+P_m-\sum_{r=0}^{m}P_rP_{(m-r)}-\xi\sum_{r=0}^{m}P_rQ_{(m-r)}\right\},\\ Q_{(m+1)}(x)&=\frac{\Gamma(m\mu+1)}{\Gamma(m\mu+\mu+1)}\left\{\frac{\partial^2 Q_m}{\partial t^2}-\lambda\sum_{r=0}^{m}P_rQ_{(m-r)},\right\},\qquad m=0,1,2,3,\ldots\end{aligned} \tag{17}$$

Using the initial condition (11), we obtain the successive solutions

$$P_0=\frac{1}{\left(e^{\sqrt{\frac{\lambda}{6}}x}+1\right)^2},\qquad Q_0=\frac{(1-\lambda)e^{\sqrt{\frac{\lambda}{6}}x}\left(e^{\sqrt{\frac{\lambda}{e}}x}+2\right)}{\xi\left(e^{\sqrt{\frac{\lambda}{e}}x}+1\right)^2},$$

$$P_1=\frac{5\lambda e^{\sqrt{\frac{\lambda}{6}}x}}{3\Gamma(\mu+1)\left(e^{\sqrt{\frac{\lambda}{e}}x}+1\right)^3},\qquad Q_1=\frac{5\lambda(1-\lambda)e^{\sqrt{\frac{\lambda}{6}}x}}{3\xi\Gamma(\mu+1)\left(e^{\sqrt{\frac{\lambda}{e}}x}+1\right)^3},$$

$$P_2=\frac{25\lambda^2 e^{\sqrt{\frac{\lambda}{6}}x}\left(2e^{\sqrt{\frac{\lambda}{6}}x}-1\right)}{18\Gamma(2\mu+1)\left(e^{\sqrt{\frac{\lambda}{e}}x}+1\right)^4},\qquad Q_2=\frac{25\lambda^2(1-\lambda)e^{\sqrt{\frac{\lambda}{6}}x}\left(2e^{\sqrt{\frac{\lambda}{6}}x}-1\right)}{18\xi\Gamma(\mu+1)\left(e^{\sqrt{\frac{\lambda}{e}}x}+1\right)^4},$$

$$P_3 = -\frac{25\lambda^3\Gamma(2\mu+1)e^{\sqrt{\frac{\lambda}{6}}x}}{9\Gamma(\mu+1)^2\Gamma(3\mu+1)(e^{\sqrt{\frac{\lambda}{6}}x}+1)^6} - \frac{25\lambda^3 e^{\sqrt{\frac{\lambda}{6}}x}(15e^{\sqrt{\frac{2\lambda}{3}}x} - 20e^{\sqrt{\frac{2\lambda}{2}}x} + 6e^{\sqrt{\frac{\lambda}{6}}x} - 5)}{108\Gamma(3\mu+1)(e^{\sqrt{\frac{\lambda}{6}}x}+1)^6},$$

$$Q_3 = -\frac{25\lambda^3(\lambda-1)\Gamma(2\mu+1)e^{\sqrt{\frac{\lambda}{6}}x}}{9\xi\Gamma(\mu+1)^2\Gamma(3\mu+1)(e^{\sqrt{\frac{\lambda}{6}}x}+1)^6} - \frac{25\lambda^3(\lambda-1)e^{\sqrt{\frac{\lambda}{6}}x}(15e^{\sqrt{\frac{2\lambda}{3}}x} - 20e^{\sqrt{\frac{2\lambda}{2}}x} + 6e^{\sqrt{\frac{\lambda}{6}}x} - 5)}{108\Gamma(3\mu+1)(e^{\sqrt{\frac{\lambda}{6}}x}+1)^6},$$

$$P_4 = -\frac{25\lambda^4\Gamma(2\mu+1)e^{\sqrt{\frac{2\lambda}{3}}x}(11e^{\sqrt{\frac{2\lambda}{3}}x} - 5e^{\sqrt{\frac{\lambda}{6}}x} - 1)}{27\Gamma(\mu+1)^2\Gamma(4\mu+1)(e^{\sqrt{\frac{\lambda}{6}}x}+1)^8} - \frac{125\lambda^4\Gamma(3\mu+1)e^{\sqrt{\frac{2\lambda}{3}}x}(2e^{\sqrt{\frac{\lambda}{6}}x} - 1)}{27\Gamma(\mu+1)^2\Gamma(2\mu+1)\Gamma(4\mu+1)(e^{\sqrt{\frac{\lambda}{6}}x}+1)^7}$$
$$+\frac{25\lambda^4\Gamma e^{\sqrt{\frac{2\lambda}{3}}x}(124e^{\sqrt{\frac{2\lambda}{3}}x} + 100e^{2\sqrt{\frac{2\lambda}{3}}x} + 85e^{\sqrt{\frac{\lambda}{6}}x} - 4)}{324\Gamma(4\mu+1)^2(e^{\sqrt{\frac{\lambda}{6}}x}+1)^8} - \frac{625\lambda^4 e^{\sqrt{\frac{\lambda}{6}}x}(17e^{2\sqrt{\frac{2\lambda}{3}}x} + 1)}{648\Gamma(4\mu+1)^2(e^{\sqrt{\frac{\lambda}{6}}x}+1)^8}$$

$$Q_4 = -\frac{25\lambda^4(\lambda-1)\Gamma(2\mu+1)e^{\sqrt{\frac{2\lambda}{3}}x}(11e^{\sqrt{\frac{2\lambda}{3}}x} - 5e^{\sqrt{\frac{\lambda}{6}}x} - 1)}{27\xi\Gamma(\mu+1)^2\Gamma(4\mu+1)(e^{\sqrt{\frac{\lambda}{6}}x}+1)^8} - \frac{125\lambda^4(\lambda-1)\Gamma(3\mu+1)e^{\sqrt{\frac{2\lambda}{3}}x}(2e^{\sqrt{\frac{\lambda}{6}}x} - 1)}{27\xi\Gamma(\mu+1)^2\Gamma(2\mu+1)\Gamma(4\mu+1)(e^{\sqrt{\frac{\lambda}{6}}x}+1)^7}$$
$$+\frac{25\lambda^4(\lambda+1)e^{\sqrt{\frac{2\lambda}{3}}x}(124e^{\sqrt{\frac{2\lambda}{3}}x} + 100e^{2\sqrt{\frac{2\lambda}{3}}x} + 85e^{\sqrt{\frac{\lambda}{6}}x} - 4)}{32\xi\Gamma(4\mu+1)(e^{\sqrt{\frac{\lambda}{6}}x}+1)^8} - \frac{625\lambda^4(\lambda-1)e^{\sqrt{\frac{\lambda}{6}}x}(17e^{2\sqrt{\frac{2\lambda}{3}}x} + 1)}{648\xi\Gamma(4\mu+1)(e^{\sqrt{\frac{\lambda}{6}}x}+1)^8}.$$

Similar expression for $P_m(x,t)$ and $Q_m(x,t)$ respectively for $m = 5, 6, 7, \ldots$ can be achieved. Then, for system of (14), the FRDTM series solution is presented by (12).

■

Example 2

Consider the nonlinear TFB-Z system at $\gamma = \lambda$ and $\beta = 1$, then (7) reduce to

$$\begin{cases} \dfrac{\partial^\mu p}{\partial t^\mu} = \dfrac{\partial^2 p}{\partial t^2} + \xi q + p - p^2 - \xi pq, & 0 < \mu \le 1, \\ \dfrac{\partial^\mu q}{\partial t^\mu} = \dfrac{\partial^2 p}{\partial t^2} + \lambda q - \lambda pq, & \end{cases} \tag{18}$$

with the initial condition

$$P(x,0) = \frac{1}{(e^{\sqrt{\frac{\lambda}{6}}x}+1)^2}, \qquad q(x,0) = \frac{\lambda-1}{\xi(e^{\sqrt{\frac{\lambda}{6}}x}+1)^2}. \tag{19}$$

The exact solution of (18) when $\mu = 1$ is

$$P(x,t) = \frac{e^{\frac{5\lambda}{2}t}}{\left(e^{\sqrt{\frac{\lambda}{6}}x} + e^{\frac{5\lambda}{2}t}\right)^2}, \qquad q(x,t) = \frac{(\lambda-1)e^{\frac{5\lambda}{2}t}}{\xi\left(e^{\sqrt{\frac{\lambda}{6}}x} + e^{\frac{5\lambda}{6}t}\right)^2}, \tag{20}$$

where $\lambda \neq 1$ is a positive parameter. ■

FRDTM Solution:

From (10) with $\gamma = \lambda$ and $\beta = 1$, we have

$$
\begin{aligned}
P_{(m+1)}(x) &= \frac{\Gamma(m\mu+1)}{\Gamma(m\mu+\mu+1)}\left\{\frac{\partial^2 P_m}{\partial t^2} + \xi Q_m + P_m - \sum_{r=0}^{m} P_r P_{(m-r)} - \xi \sum_{r=0}^{m} P_r Q_{(m-r)}\right\},\\
Q_{(m+1)}(x) &= \frac{\Gamma(m\mu+1)}{\Gamma(m\mu+\mu+1)}\left\{\frac{\partial^2 Q_m}{\partial t^2} + \lambda Q_m - \lambda \sum_{r=0}^{m} P_r Q_{(m-r)}\right\}.
\end{aligned}
\tag{21}
$$

Using the initial condition (11), we obtain the successive solutions

$$
P_0 = \frac{1}{\left(e^{\sqrt{\frac{\lambda}{6}}x}+1\right)^2}, \qquad Q_0 = \frac{(1-\lambda)}{\xi\left(e^{\sqrt{\frac{\lambda}{e}}x}+1\right)^2},
$$

$$
P_1 = \frac{5\lambda e^{\sqrt{\frac{\lambda}{6}}x}}{3\Gamma(\mu+1)\left(e^{\sqrt{\frac{\lambda}{e}}x}+1\right)^3}, \qquad Q_1 = \frac{5\lambda(\lambda-1)e^{\sqrt{\frac{\lambda}{6}}x}}{3\xi\Gamma(\mu+1)\left(e^{\sqrt{\frac{\lambda}{e}}x}+1\right)^3},
$$

$$
P_2 = \frac{25\lambda^2 e^{\sqrt{\frac{\lambda}{6}}x}\left(2e^{\sqrt{\frac{\lambda}{6}}x}-1\right)}{18\Gamma(2\mu+1)\left(e^{\sqrt{\frac{\lambda}{e}}x}+1\right)^4}, \qquad Q_2 = \frac{25\lambda^2(1-\lambda)e^{\sqrt{\frac{\lambda}{6}}x}\left(2e^{\sqrt{\frac{\lambda}{6}}x}-1\right)}{18\xi\Gamma(2\mu+1)\left(e^{\sqrt{\frac{\lambda}{e}}x}+1\right)^4},
$$

$$
P_3 = -\frac{25\lambda^3\Gamma(2\mu+1)e^{\sqrt{\frac{2\lambda}{3}}x}}{9\Gamma(\mu+1)^2\Gamma(3\mu+1)(e^{\sqrt{\frac{\lambda}{6}}x}+1)^6} - \frac{25\lambda^3 e^{\sqrt{\frac{\lambda}{6}}x}(15e^{\sqrt{\frac{2\lambda}{3}}x}-20e^{\sqrt{\frac{3\lambda}{2}}x}+6e^{\sqrt{\frac{\lambda}{6}}x}-5)}{108\Gamma(3\mu+1)(e^{\sqrt{\frac{\lambda}{6}}x}+1)^6},
$$

$$
Q_3 = -\frac{25\lambda^3(\lambda-1)\Gamma(2\mu+1)e^{\sqrt{\frac{2\lambda}{3}}x}}{9\xi\Gamma(\mu+1)^2\Gamma(3\mu+1)(e^{\sqrt{\frac{\lambda}{6}}x}+1)^6} - \frac{25\lambda^3(\lambda-1)e^{\sqrt{\frac{\lambda}{6}}x}(15e^{\sqrt{\frac{2\lambda}{3}}x}-20e^{\sqrt{\frac{3\lambda}{2}}x}+6e^{\sqrt{\frac{\lambda}{6}}x}-5)}{108\Gamma(3\mu+1)(e^{\sqrt{\frac{\lambda}{6}}x}+1)^6},
$$

$$
\begin{aligned}
P_4 = &-\frac{25\lambda^4(\lambda-1)\Gamma(2\mu+1)e^{\sqrt{\frac{2\lambda}{3}}x}(11e^{\sqrt{\frac{2\lambda}{3}}x}-5e^{\sqrt{\frac{\lambda}{6}}x}-1)}{27\Gamma(\mu+1)^2\Gamma(4\mu+1)(e^{\sqrt{\frac{\lambda}{6}}x}+1)^8} - \frac{125\lambda^4(\lambda-1)\Gamma(3\mu+1)e^{\sqrt{\frac{2\lambda}{3}}x}(2e^{\sqrt{\frac{\lambda}{6}}x}-1)}{27\Gamma(\mu+1)\Gamma(2\mu+1)\Gamma(4\mu+1)(e^{\sqrt{\frac{\lambda}{6}}x}+1)^7}\\
&+\frac{25\lambda^4 e^{\sqrt{\frac{2\lambda}{3}}x}(124e^{\sqrt{\frac{2\lambda}{3}}x}+100e^{2\sqrt{\frac{2\lambda}{3}}x}+85e^{\sqrt{\frac{\lambda}{6}}x}-4)}{324\Gamma(4\mu+1)(e^{\sqrt{\frac{\lambda}{6}}x}+1)^8} - \frac{625\lambda^4 e^{\sqrt{\frac{\lambda}{6}}x}(17e^{2\sqrt{\frac{2\lambda}{3}}x}+1)}{648\Gamma(4\mu+1)(e^{\sqrt{\frac{\lambda}{6}}x}+1)^8},
\end{aligned}
$$

$$Q_4 = -\frac{25\lambda^4(\lambda-1)\Gamma(2\mu+1)e^{\sqrt{\frac{2\lambda}{3}}x}(11e^{\sqrt{\frac{2\lambda}{3}}x} - 5e^{\sqrt{\frac{\lambda}{6}}x} - 1)}{27\xi\Gamma\left(\mu+1\right)^2\Gamma\left(4\mu+1\right)(e^{\sqrt{\frac{\lambda}{6}}x}+1)^8} - \frac{125\lambda^4(\lambda-1)\Gamma(3\mu+1)e^{\sqrt{\frac{2\lambda}{3}}x}(2e^{\sqrt{\frac{\lambda}{6}}x} - 1)}{27\xi\Gamma\left(\mu+1\right)\Gamma(2\mu+1)\Gamma\left(4\mu+1\right)(e^{\sqrt{\frac{\lambda}{6}}x}+1)^7}$$
$$+\frac{25\lambda^4(\lambda-1)e^{\sqrt{\frac{2\lambda}{3}}x}(124e^{\sqrt{\frac{2\lambda}{3}}x} + 100e^{2\sqrt{\frac{2\lambda}{3}}x} + 85e^{\sqrt{\frac{\lambda}{6}}x} - 4)}{324\xi\Gamma\left(4\mu+1\right)(e^{\sqrt{\frac{\lambda}{6}}x}+1)^8} - \frac{625\lambda^4(\lambda-1)e^{\sqrt{\frac{\lambda}{6}}x}(17e^{2\sqrt{\frac{2\lambda}{3}}x}+1)}{648\xi\Gamma\left(4\mu+1\right)(e^{\sqrt{\frac{\lambda}{6}}x}+1)^8}.$$

Similar expression for P_m and Q_m respectively for $m = 5, 6, 7, L$ can be achieved. Then, for system of (14) with initial condition (16), the FRDTM series solution is presented by (12).

■

Remark

Assume that $q = \frac{\lambda - 1}{\xi} p$ in (18), then the system reduces to the time-fractional Fisher's equation

$$\frac{\partial^\mu p}{\partial t^\mu} = \frac{\partial^2 p}{\partial t^2} + \lambda p - \lambda p^2, \tag{22}$$

which represents a model for the propagation of a mutant gene (Fisher, 937), where $p(x,t)$ denotes the population density, $p(1-p)$ stand for the population supply due to births and deaths and the birth rate λ (Fisher, 1937). The exact solution of (22) for the case when $\mu = 1$ is obtained from (20) and is given for positive parameter λ as

$$p(x,t) = \frac{e^{\frac{5\lambda}{2}t}}{\left(e^{\sqrt{\frac{\lambda}{6}}x} + e^{\frac{5\lambda}{2}t}\right)^2} = \frac{1}{4}\left(\tanh\left(\sqrt{\frac{\lambda}{24}}x - \frac{5\lambda}{12}t\right) - 1\right)^2. \tag{23}$$

■

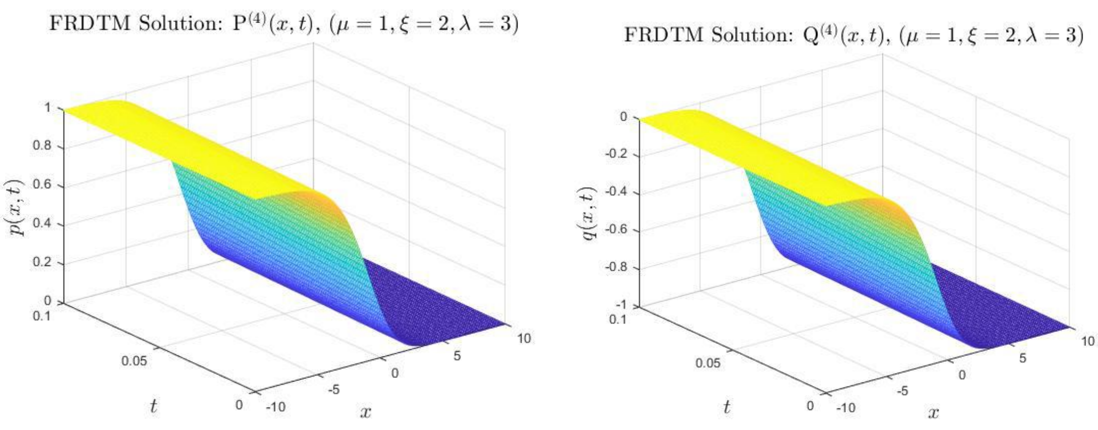

Figure 1: The among FRDTM solution $P^{(4)}(x, t)$ and $Q^{(4)}(x, t)$ for Example 1.

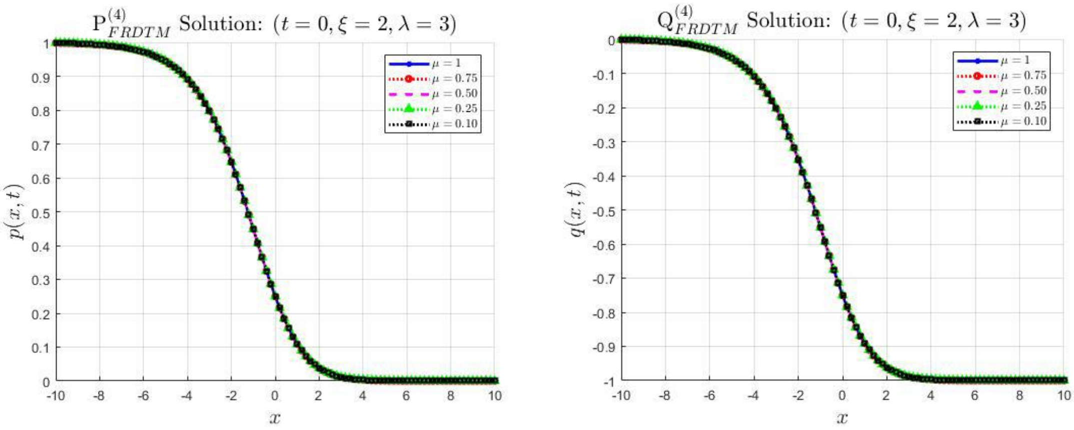

Figure 2: Solution profile with different μ values when $t = 0$ for Example 1.

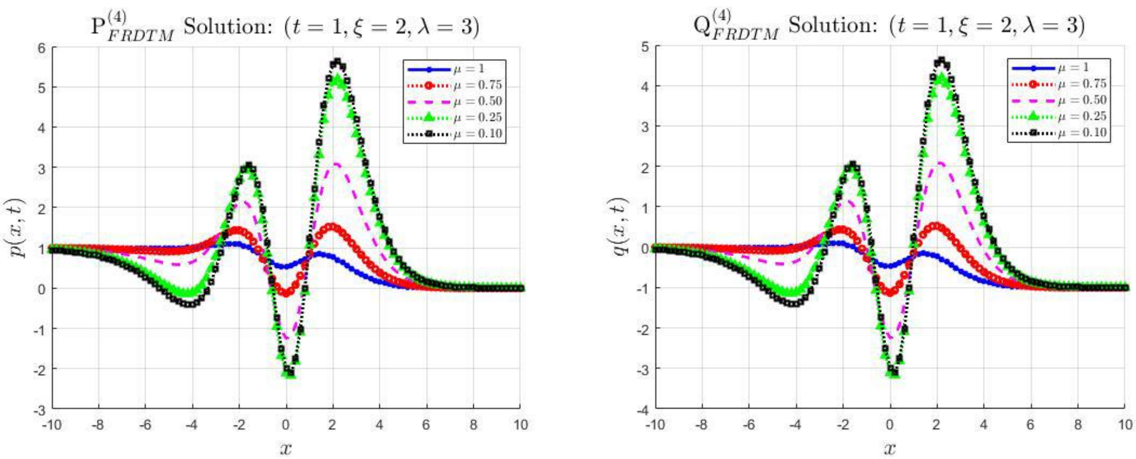

Figure 3: Solution profile with different μ values when $t = 1$ for Example 1.

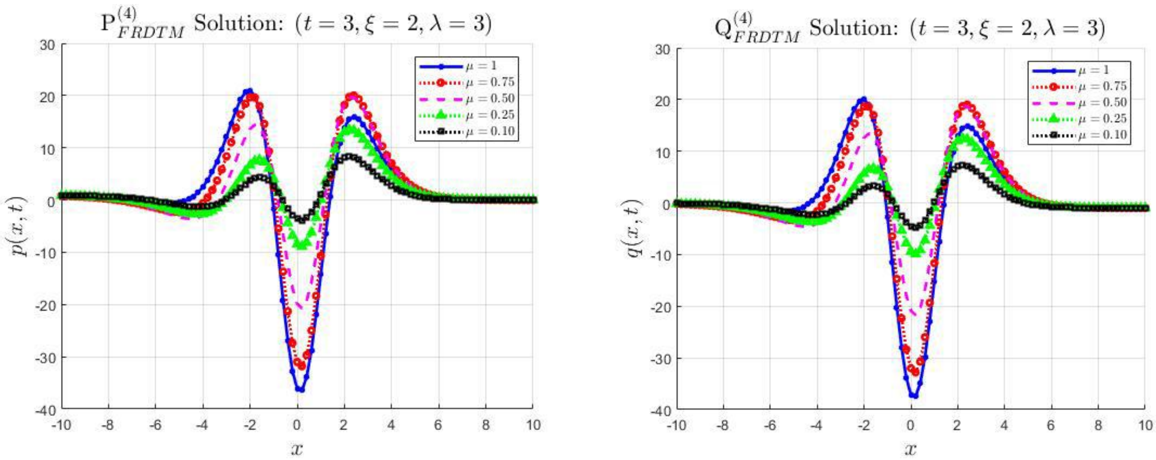

Figure 4: Solution profile with different μ values when $t = 3$ for Example 1.

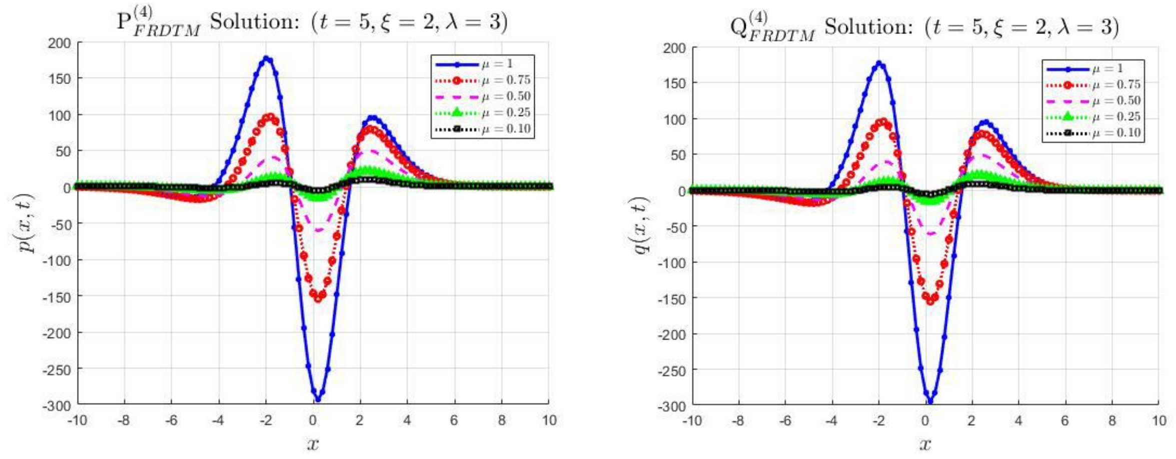

Figure 5: Solution profile with different μ values when $t = 5$ for Example 1.

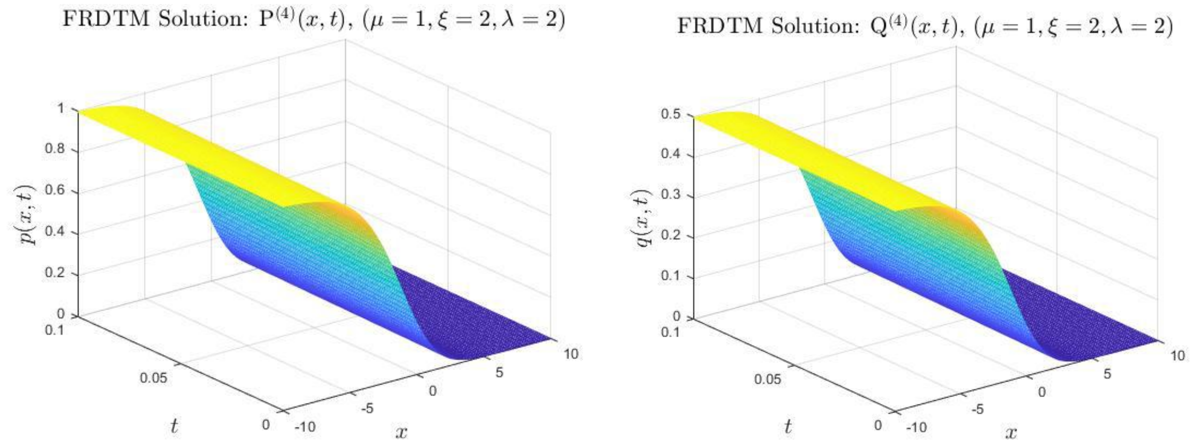

Figure 6: The among FRDTM solution $P^{(4)}(x, t)$ and $Q^{(4)}(x, t)$ for Example 2.

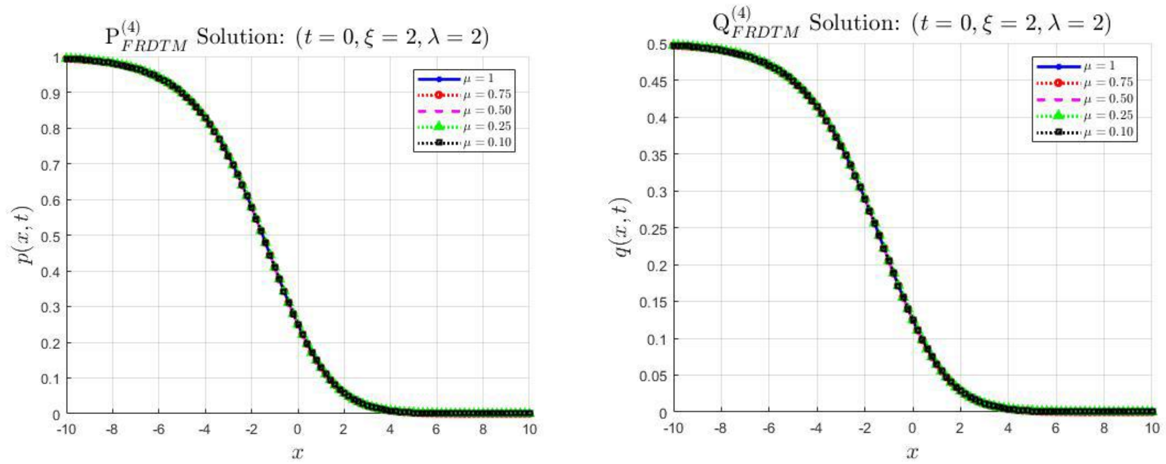

Figure 7: Solution profile with different μ values when $t = 0$ for Example 2.

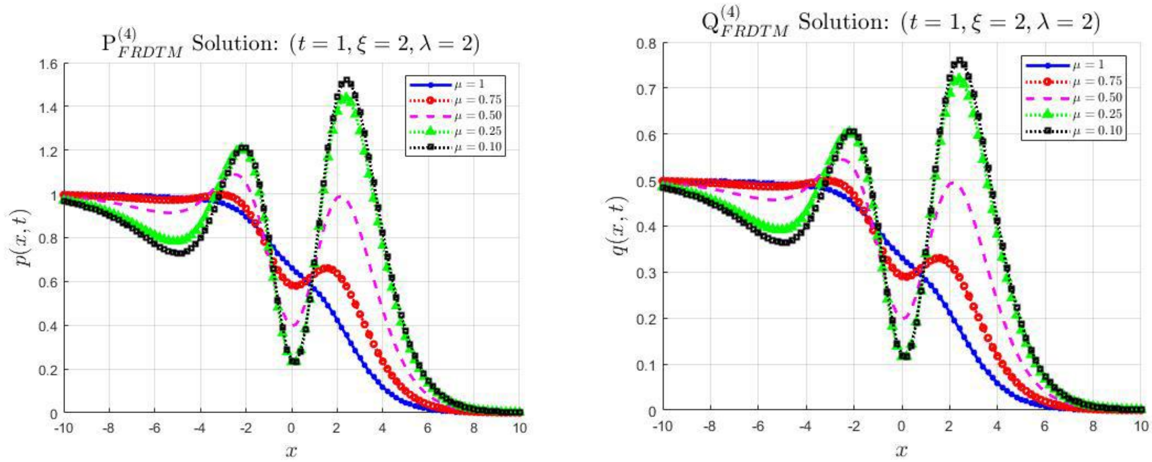

Figure 8: Solution profile with different μ values when $t = 1$ for Example 2.

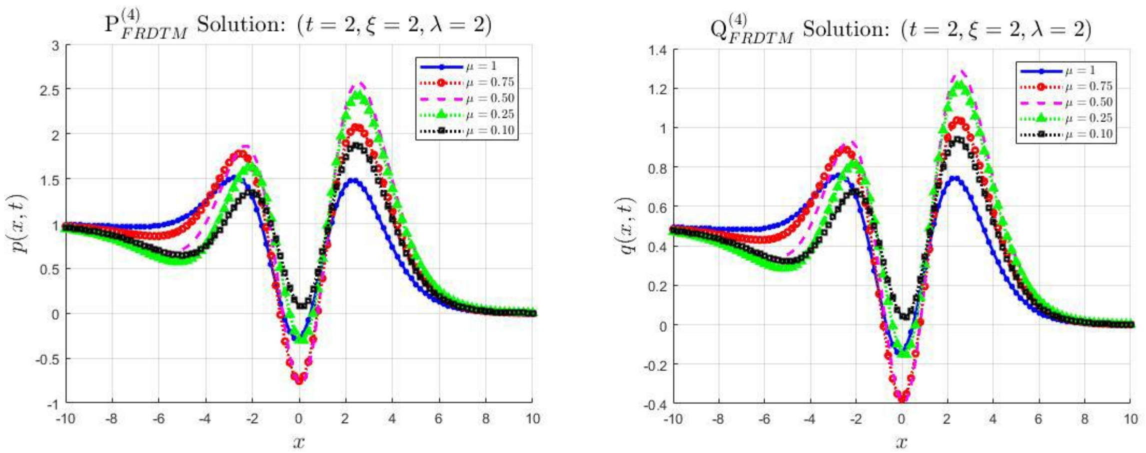

Figure 9: Solution profile with different μ values when $t = 2$ for Example 2.

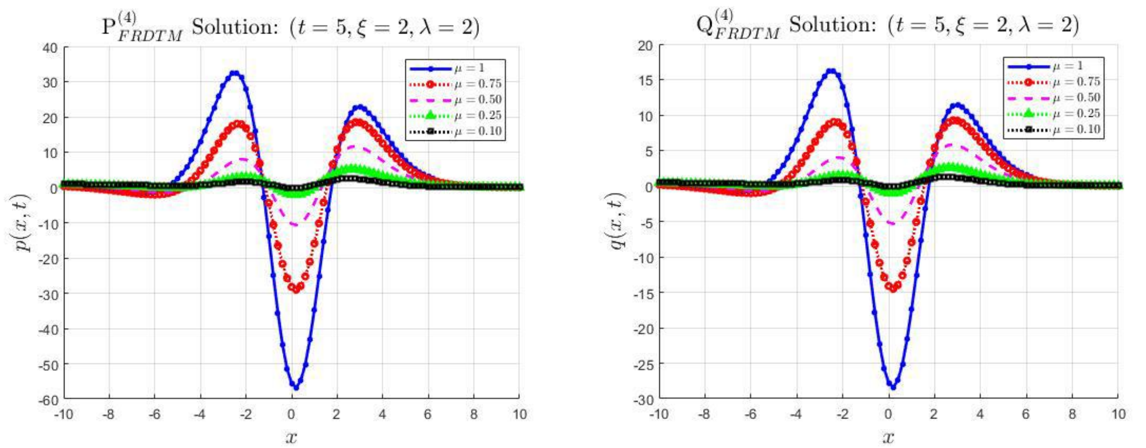

Figure 10: Solution profile with different μ values when $t = 5$ for Example 2.

■

Conclusion

Using reliable technique, namely, the fractional reduced differential transform method (FRDT) solved the time-fractional Belousov-Zhabotinsky system. The outcomes of this research give the very effective and accurate (FRDT) method which had a wide ranging feasibility. Therefore, it can solve a lot of strong nonlinear fractional and classical PDEs that arisen in physics, chemistry, biology, mathematics, engineering, and so on.

Acknowledgements

The author is grateful to the members of the UMRJ Editorial Board for their valuable comments and suggestions on the manuscript. The author acknowledges to Professor Dr Khin Myo Aye and Professor Dr Khin Phyu Phyu Htoo for their encouragements on doing research.

References

Adomian, G. (1994), Solving Frontier Problems of Physics: The Decomposition Method", Springer, Netherlands.

Baleanu, D., Guvenc, Z. B. and Machado, J. T. (2010), "New Trends in Nanotechnology and Fractional Calculus Applications", Springer.

Baleanu, D., Wu, G.C. and Zeng, S.D. (2017), "Chaos Analysis and Asymptotic Stability of Generalized Caputo Fractional Differential Equations", Chaos Solitons Fractals 102, pp. 99-105.

Caputo, M. (1969), "Elasticita e Dissipazione", Zani-Chelli, Bologna.

Chen, D., Chen, Y. and Xue, D. (2013), "Three Fractional-Order TV-Models for Image De-Noising", J. Comput. Inf. Sys. 9 (12), pp. 4773-80.

Das, S. (2009), "Analytical Solution of a Fractional Diffusion Equation by Variational Ieration Method", Comput Math. Appl. 57, pp. 483-487.

Dhaigude, C. D. and Nikam, V. R. (2012), " Solution of Fractional Partial Differential Equations Using Iterative Method", Fractional Calculus and Applied Analysis 15(4), pp. 684-699.

Eltayeb, H. and Kilicman, A. (2012), "Application of Sumudu Decomposition Method to Solve Nonlinear System of Partial Differential Equations", Abstract and Applied Analysis 2012, pp.1-13.

Fisher, R. A. (1937), " The Wave of Advance of Advantageous Genes", Ann. Eugen. 7(4), pp. 355-369.

Gibbs, R. G. (1980), "Traveling Waves in the Belousov-Zhabotinskii Reaction", SIAM Journal on Applied Mathematics 38(3), pp. 422-444.

He, J. H. (1998), "Approximate Analytical Solution for Seepage Flow with Fractional Derivatives in Porous Media, Comput. Meth Appl. Mech. Eng 167, pp. 57-68.

He, J. H. (1999), " Homotopy Merturbation Technique", Comput. Meth Appl Mech. Eng 178, pp. 257-62.

He, J. H. (2003), " Homotopy Perturbation Method: a New Nonlinear Analytical Technique", Appl. Math Comput. 135, pp. 73-9.

Hilfer, R. and Anton, L. (1995), "Fractional Master Equations and Fractal Time Random Walks", Physical Review E, 51, R848-R851

Jaradat, A., Noorani, M. S. M., Alquran, M. and Jaradat, H. M. (2018), "Numerical Investigations for Time-Fractional Nonlinear Model Arise in Physics", Results in Physics, 8, pp. 1034-1037

Joseph, K., Lim, S. C. and Metzler, R. (2012), "Fractional Dynamics: Recent Advances", World Scientific, Singapore.

Khuri, S. A. (2001), "A Laplace Decomposition Algorithm Applied to Class of Nonlinear Differential Equations", J. Math Appl. 1(4), pp. 141-155.

Kilbas, A. A., Srivastava, H. M. and Trujillo, J. J. (2006), "Theory and Applications of Fractional Differential Equations", North-Holland Mathematics Studies. Vol. 204, Elsevier Science B. V., Amsterdam, the Netherlands.

Liao, S. J., (1998), " Homotopy Analysis Method: A New Analytic Method for Nonlinear Problems", Appl. Math Mech, 19, pp. 957-962

Liao, S. J. (2004), " On the Homotopy Analysis Method for Nonlinear Problems", Applied Mathematics and Computation 147(2), pp. 499-513

Mainardi, F. (2010), "Fractional Calculus and Waves in Linear Viscoelasticity", Imperial College Press.

Miller, K. S. and Ross, B. (1993), "An Introduction to Fractional Calculus and Fractional Differential Equations", Wiley, New York

Nasrolahpour, H. (2013), "A Note on Fractional Electrodynamics", Commun. Nonlinear Sci. Numer. Simul. 18, pp. 2589-2593.

Podlubny, I. (1999), "Fractional Differential Equations", Academic Press, New York.

Senol, M., Iyiola, O. S., Dae, K. H. and Akinyemi, L. (2019), " Efficient Analytical Techniques for Solving Time-Fractional Nonlinear Coupled Jaulent-Miodek System with Energy-Dependent Schrodinger Potential", Adv. Differ. Equ., pp. 1-21.

Tarasov, V. E. (2006), "Gravitational Field of Fractal Distribution of Particles", Celestial Mechanics and Dynamical Astronomy, 94 (1), pp. 1-15.

Ye, Q. X. and Wang, M. X. (1987), "Travelling Wave Front Solutions of Noyes-Field System for Belousov-Zhabotinskii Reaction, Nonlinear Anal., 1, pp. 1289-1302.

Appendix

Appendix 1

It should be noted that the exact solution for Example 1 and Example 2 can also have the following forms:

- **For Example 1:**

$$\frac{e^{\frac{5\lambda}{2}t}}{\left(e^{\sqrt{\frac{\lambda}{6}}x}+e^{\frac{5\lambda}{6}t}\right)^2}=\frac{1}{4}\left(\tanh^2\left(\sqrt{\frac{\lambda}{24}}x-\frac{5\lambda}{12}t\right)-1\right)^2,$$

$$\frac{(1-\lambda)e^{\sqrt{\frac{\lambda}{6}}x}\left(e^{\sqrt{\frac{\lambda}{6}}x}+2e^{\frac{5\lambda}{6}t}\right)}{\xi\left(e^{\sqrt{\frac{\lambda}{6}}x}+e^{\frac{5\lambda}{6}t}\right)^2}=\frac{\lambda-1}{4\xi}\left(\tanh^2\left(\sqrt{\frac{\lambda}{24}}x-\frac{5\lambda}{12}t\right)-2\tanh\left(\sqrt{\frac{\lambda}{24}}x-\frac{5\lambda}{12}t\right)-3\right).$$

- **For Example 2:**

$$\frac{e^{\frac{5\lambda}{3}t}}{\left(e^{\sqrt{\frac{\lambda}{6}}x}+e^{\frac{5\lambda}{6}t}\right)^2}=\frac{1}{4}\left(\tanh\left(\sqrt{\frac{\lambda}{24}}x-\frac{5\lambda}{12}t\right)-1\right)^2,$$

$$\frac{(1-\lambda)e^{\sqrt{\frac{\lambda}{6}}x}\left(e^{\sqrt{\frac{\lambda}{6}}x}+2e^{\frac{5\lambda}{6}t}\right)}{\xi\left(e^{\sqrt{\frac{\lambda}{6}}x}+e^{\frac{5\lambda}{6}t}\right)^2}=\frac{\lambda-1}{4\xi}\tanh\left(\sqrt{\frac{\lambda}{24}}x-\frac{5\lambda}{12}t-1\right)^2.$$